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SOLUTION OF POISSON AND LAPLACE EQUATIONS BY QUADRILATERAL QUADRATURE ELEMENT

HONGZHI ZHONG* and YUHONG HE
Department of Civil Engineering, Tsinghua University, Beijing 100084, P.R. China

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Abstract—The Quadrature Element Method (QEM) is a new concept in numerical methods which was introduced only recently. A quadrilateral quadrature element is developed in the paper and applied to the solution of two dimensional potential problems governed by Poisson or Laplace equations. The results of three examples are in good agreement with the available solutions. It is shown that the present quadrilateral quadrature element is very efficient and effective. Most significantly, it can be employed to solve differential equations of physical problems with irregular geometry and complex boundary conditions.

1. INTRODUCTION

It is well-known that analytical solutions to various differential equations are limited to idealized situations wherein the physical domain of the problem is regular and the boundary conditions are simple. To deal with many problems in practice of complicated boundary conditions and irregular geometric shape, one has to resort to numerical methods such as the Finite Difference Method (FDM) and the Finite Element Method (FEM) which have become dominant numerical tools after decades of development. Usually, a large number of discretized points are needed in these methods to attain high accuracy of results. Consequently, the computational cost can often be prohibitive for these numerical methods. Therefore, the effort to seek alternative numerical methods which are cost-saving and efficient has never stopped. Since it was first introduced by Bellman and Casti (1971), the Differential Quadrature Method (DQM) has found many applications (Jang *et al.*, 1989) in solving directly the governing equations of engineering and mathematical physics. It has been found that the DQM is very efficient and accurate with relatively coarse mesh. However, the major drawback it suffers is that the applications are restricted to simple geometry and simple boundary conditions. To solve partial differential equations in curvilinear coordinate systems, efforts were made by Lam (1993) and subsequently by Shu *et al.* (1995). To extend the applicability of the DQM to irregular shapes, an attempt was also made recently by Zhong (1996) to map the irregular physical domains into the normalized unit square domain which is required by the DQM. In the analysis, a proper mathematical transformation needs to be conducted; meanwhile, the Taylor expansion technique is needed to remove the singularity which may appear in the inverse Jacobian matrix of the transformation. Therefore, it is still difficult to apply the transformation technique when the physical domain of a problem is arbitrary or its boundary conditions are complex.

To overcome the above-mentioned difficulty, the idea of "multi-domain differential quadrature method" was contrived by Shu *et al.* (1991, 1992). Later, the same approach was extended to slender structure components by Striz *et al.* (1994), and the Quadrature Element Method (QEM) was coined. In the method, a physical domain is divided into several subdomains, i.e., quadrature elements. On the inner points of each quadrature element, the conventional DQM deriving from the strong form of the governing equations is applied while on the common boundary of two adjacent elements, enforcement of the continuity of the function and its derivatives is required. In other words, the DQM can be regarded as the particular case of QEM, i.e., the case with one quadrature element only.

* Author to whom correspondence should be addressed.

Thus, the merits of both the FEM and the DQM are inherited by the QEM. The global coefficient matrix can be assembled by integrating the continuity conditions on each internal boundary of two adjacent quadrature elements. The QEM has been successfully used to analyze axisymmetric plates (Zhong, 1995) as well as structures made of slender members (beams, trusses and frames) (Striz *et al.*, 1994). In this paper, a quadrilateral quadrature element is developed and used to deal with two dimensional potential problems governed by Poisson or Laplace equation. Three numerical examples are given to demonstrate the feasibility and high accuracy of the quadrilateral quadrature element. More significantly, as shown by the three examples, the present element vitalizes the newly developed quadrature element concept and enables it to be applicable to various physical and engineering problems of complicated boundary conditions and geometric shapes. It is also shown that the QEM is an alternative powerful numerical tool to the FEM and the FDM.

2. DIFFERENTIAL QUADRATURE FORMULATION OF POTENTIAL PROBLEMS

The differential equation of a two dimensional potential problem is given as follows

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = f(x, y), \quad (x, y) \in \Omega \tag{1}$$

where x, y are space variables, Ω is the physical domain, $f(x, y)$ is a known function defined in Ω, $u(x, y)$ the function to be solved. The general boundary condition can be expressed as

$$a_1 u + a_2 \frac{\partial u}{\partial n} = \bar{u}, \quad (x, y) \in \Gamma \tag{2}$$

where Γ is the boundary of the domain, $\bar{u}$ is a known constant. Equation (2) becomes a Dirichlet type boundary condition for $a_2 = 0$ while Neumann boundary condition corresponds to the case when $a_1 = 0$. According to the requirement of the QEM, the following general transformation is introduced to map the physical domain (x, y) into the normalized computational domain (ξ, η).

$$\begin{cases} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{cases} \quad 0 \leqslant \xi, \eta \leqslant 1 \tag{3}$$

where x and y are space variables defined on the physical domain of the problem of concern. The Jacobian matrix of the transformation is

$$[J] = \begin{bmatrix} x_\xi & y_\xi \\ x_\eta & y_\eta \end{bmatrix} \tag{4}$$

Its inverse matrix is given as follows

$$\begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} = [J]^{-1} = \frac{1}{|J|} \begin{bmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{bmatrix}, \quad |J| = x_\xi y_\eta - x_\eta y_\xi \tag{5}$$

where $|J|$ is the determinant of the Jacobian matrix. Thus, the transformation of the first order derivatives is

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{6}$$

Then, eqn (1) becomes

$$F_1(\xi,\eta)\frac{\partial^2 u}{\partial \xi^2} + F_2(\xi,\eta)\frac{\partial^2 u}{\partial \xi \partial \eta} + F_3(\xi,\eta)\frac{\partial^2 u}{\partial \eta^2} + F_4(\xi,\eta)\frac{\partial u}{\partial \xi} + F_5(\xi,\eta)\frac{\partial u}{\partial \eta} = F_0(\xi,\eta) \tag{7}$$

where

$$\begin{aligned} F_1(\xi,\eta) &= \left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2 \\ F_2(\xi,\eta) &= 2\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\right) \\ F_3(\xi,\eta) &= \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 \\ F_4(\xi,\eta) &= \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \\ F_5(\xi,\eta) &= \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \\ F_0(\xi,\eta) &= f(x(\xi,\eta), y(\xi,\eta)) \end{aligned} \tag{8}$$

The transformation needs to be conducted when derivative appears in boundary conditions. Suppose the unit normal at a boundary is $\mathbf{n} = (\alpha, \beta)$, where α and β are the direction cosines of $\mathbf{n}$. The normal derivative in eqn (2) can then be written as

$$\frac{\partial u}{\partial n} = [\alpha, \beta] \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = [\alpha, \beta][J]^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \frac{1}{|J|}\left[(\alpha y_\eta - \beta x_\eta)\frac{\partial u}{\partial \xi} - (\alpha y_\xi - \beta x_\xi)\frac{\partial u}{\partial \eta}\right] \tag{9}$$

As discussed before, the QEM is based on the DQM, i.e., the partial derivatives of a function with respect to a space variable at a given point is approximated by a weighted linear sum of the function values at all discrete points in the quadrature element. The application of differential quadrature approximation to eqn (7) leads to a set of linear algebraic equations

$$F_1 \sum_{k=1}^{N_\xi} W_{ik}^{(2)} u_{kj} + F_2 \sum_{m=1}^{N_\eta} W_{jm}^{(1)} \sum_{k=1}^{N_\xi} W_{ik}^{(1)} u_{km} + F_3 \sum_{m=1}^{N_\eta} W_{jm}^{(2)} u_{im} + F_4 \sum_{k=1}^{N_\xi} W_{ik}^{(1)} u_{kj}$$
$$+ F_5 \sum_{m=1}^{N_\eta} W_{jm}^{(1)} u_{im} = F_0 \quad (i = 2, 3, \ldots, N_\xi - 1; \quad j = 2, 3, \ldots, N_\eta - 1) \tag{10}$$

where N_ξ and N_η are the number of grids in the two directions of the normalized domain which can be different. They are usually taken as same N since it is easy for quadrature elements to match each other. $W_{ij}^{(n)}$ are the weighting coefficients related to the function values. In this paper, the Lagrangian interpolation formula (Shu and Richards, 1993) is used to determine the weighting coefficients. Its significant advantage is that it does not require to solve simultaneous equations, since the weighting coefficients are expressed in explicit forms. In addition, the weighting coefficients of high order derivatives have recurrent relations with those of the lower order derivatives. The expressions of the weighting coefficients in the ξ direction are as follows

$$W_{ij}^{(1)} = \begin{cases} \dfrac{M^{(1)}(\xi_i)}{(\xi_i-\xi_j)M^{(1)}(\xi_j)}, & i \neq j \\ -\displaystyle\sum_{k=1,k\neq i}^{N} W_{ik}^{(1)}, & i = j \end{cases} \tag{11}$$

$$W_{ij}^{(m)} = \begin{cases} m\left[W_{ii}^{(m-1)}W_{ij}^{(1)} - \dfrac{W_{ij}^{(m-1)}}{(\xi_i-\xi_j)}\right], & i \neq j \\ -\displaystyle\sum_{k=1,k\neq i}^{N} W_{ik}^{(m)}, & i = j \end{cases} \tag{12}$$

where

$$M(\xi) = \prod_{j=1}^{N}(\xi-\xi_j)$$

$$M^{(1)}(\xi_k) = \prod_{j=1,j\neq k}^{N}(\xi_k-\xi_j), \quad k = 1,2,\ldots,N \tag{13}$$

In the η direction, the formulas for the weighting coefficients are similar to the above.

Usually, the governing equations are employed at all inner grid points. The differential quadrature approximation of eqn (2) results in a series of algebraic equations for the mesh points on the four boundaries of the (ξ,η) domain, i.e., $\xi = 0$, $\xi = 1$; $\eta = 0$, $\eta = 1$. For a quadrature element with $N \times N$ grid points, there are $(N-2)\times(N-2)$ inner grid points at which the governing equations are applied. On the four sides of an element, there are $4(N-1)$ grid points at which boundary conditions or continuity conditions are prescribed. Values of $u(\xi,\eta)$ and its derivatives will be obtained by solving the resulting set of simultaneous algebraic equations.

3. QUADRILATERAL QUADRATURE ELEMENT FOR POTENTIAL PROBLEMS

3.1. *Element formulation*

A quadrilateral element in the global coordinate system (x,y) and its unit square counterpart in the local coordinate system (ξ,η) are shown in Fig. 1. The transformation relations are given as follows.

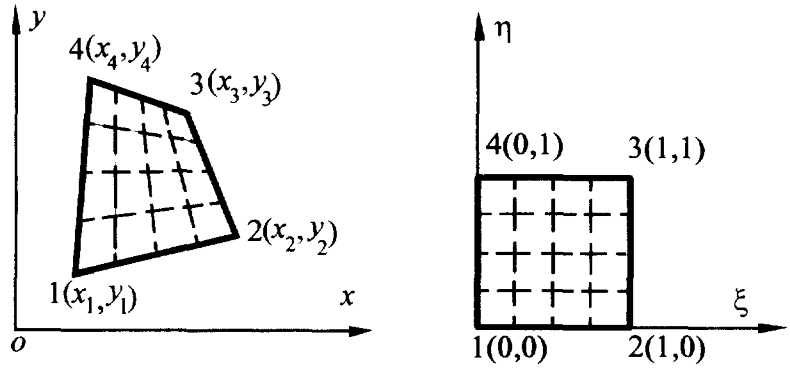

Fig. 1. A quadrilateral element and its unit square counterpart.

$$\begin{cases} x = \sum_{i=1}^{4} N_i x_i \\ y = \sum_{i=1}^{4} N_i y_i \end{cases} \tag{14}$$

where

$$\begin{aligned} N_1(\xi,\eta) &= (1-\xi)(1-\eta) \\ N_2(\xi,\eta) &= \xi(1-\eta) \\ N_3(\xi,\eta) &= \xi\eta \\ N_4(\xi,\eta) &= (1-\xi)\eta \end{aligned} \tag{15}$$

The Jacobian matrix of the above transformation is given in eqn (4) in which

$$\begin{aligned} x_\xi &= -x_1+x_2+\eta(x_1-x_2+x_3-x_4) \\ y_\xi &= -y_1+y_2+\eta(y_1-y_2+y_3-y_4) \\ x_\eta &= -x_1+x_4+\xi(x_1-x_2+x_3-x_4) \\ y_\eta &= -y_1+y_4+\xi(y_1-y_2+y_3-y_4) \end{aligned} \tag{16}$$

From eqn (5), the element of the inverse Jacobian matrix can also be expressed as functions of ξ, η, i.e.,

$$\xi_x = y_\eta/|J|, \quad \eta_x = -y_\xi/|J|, \quad \xi_y = -x_\eta/|J|, \quad \eta_y = x_\xi/|J| \tag{17}$$

where

$$|J| = x_\xi y_\eta - y_\xi x_\eta \tag{18}$$

Therefore,

$$\frac{\partial}{\partial x} = \frac{1}{|J|}\left(y_\eta \frac{\partial}{\partial \xi} - y_\xi \frac{\partial}{\partial \eta}\right), \quad \frac{\partial}{\partial y} = \frac{1}{|J|}\left(-x_\eta \frac{\partial}{\partial \xi} + x_\xi \frac{\partial}{\partial \eta}\right)$$

and

$$\begin{aligned} \frac{\partial^2 \xi}{\partial^2 x} &= \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) = \frac{1}{|J|}\left(y_\eta \frac{\partial}{\partial \xi} - y_\xi \frac{\partial}{\partial \eta}\right)\left(\frac{y_\eta}{|J|}\right) \\ &= \frac{1}{|J|^2}\left(y_\eta y_{\xi\eta} - \frac{y_\eta^2}{|J|}\frac{\partial}{\partial \xi}(|J|) - y_\xi y_{\eta\eta} + \frac{y_\xi y_\eta}{|J|}\frac{\partial}{\partial \eta}(|J|)\right) \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial^2 \xi}{\partial^2 y} &= \frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial y}\right) = \frac{1}{|J|}\left(-x_\eta \frac{\partial}{\partial \xi} + x_\xi \frac{\partial}{\partial \eta}\right)\left(-\frac{x_\eta}{|J|}\right) \\ &= \frac{1}{|J|^2}\left(x_\eta x_{\xi\eta} - \frac{x_\eta^2}{|J|}\frac{\partial}{\partial \xi}(|J|) - x_\xi x_{\eta\eta} + \frac{x_\xi x_\eta}{|J|}\frac{\partial}{\partial \eta}(|J|)\right) \end{aligned} \tag{20}$$

Differentiation of eqn (16) yields

$$x_{\xi\xi} = x_{\eta\eta} = y_{\xi\xi} = y_{\eta\eta} = 0 \tag{21}$$

In combination of eqn (21), the differentiation of eqn (18) leads to

$$\frac{\partial}{\partial \xi}(|J|) = x_{\xi} y_{\xi\eta} - y_{\xi} x_{\xi\eta} \tag{22}$$

$$\frac{\partial}{\partial \eta}(|J|) = x_{\xi\eta} y_{\eta} - y_{\xi\eta} x_{\eta} \tag{23}$$

where

$$x_{\xi\eta} = x_1 - x_2 + x_3 - x_4 \tag{24}$$

$$y_{\xi\eta} = y_1 - y_2 + y_3 - y_4 \tag{25}$$

Substitution of eqns (21)–(23) into (19) and (20) results in

$$\frac{\partial^2 \xi}{\partial^2 x} = \frac{y_\eta}{|J|^2}\left[\left(1 - \frac{x_\xi y_\eta + y_\xi x_\eta}{|J|}\right) y_{\xi\eta} + \left(\frac{2 y_\xi y_\eta}{|J|}\right) x_{\xi\eta}\right] \tag{26}$$

$$\frac{\partial^2 \xi}{\partial^2 y} = \frac{x_\eta}{|J|^2}\left[\left(1 + \frac{x_\xi y_\eta + y_\xi x_\eta}{|J|}\right) x_{\xi\eta} - \left(\frac{2 x_\xi x_\eta}{|J|}\right) y_{\xi\eta}\right] \tag{27}$$

In the same way, the following relations can be obtained.

$$\frac{\partial^2 \eta}{\partial^2 x} = \frac{y_\xi}{|J|^2}\left[\left(1 + \frac{x_\xi y_\eta + y_\xi x_\eta}{|J|}\right) y_{\xi\eta} - \left(\frac{2 y_\xi y_\eta}{|J|}\right) x_{\xi\eta}\right] \tag{28}$$

$$\frac{\partial^2 \eta}{\partial^2 y} = \frac{x_\xi}{|J|^2}\left[\left(1 - \frac{x_\xi y_\eta + y_\xi x_\eta}{|J|}\right) x_{\xi\eta} + \left(\frac{2 x_\xi x_\eta}{|J|}\right) y_{\xi\eta}\right] \tag{29}$$

thus, all the transformation functions for eqn (7), i.e., $F_i(\xi, \eta)$ $(i = 0, 1, \ldots, 5)$ in eqn (8), can be expressed as functions of ξ and η. It is worthwhile to point out that the above mathematical manipulations are valid only when $|J| \neq 0$, requiring that any interior angle of a quadrilateral quadrature element must be less than 180 degrees.

3.2. *Treatment of boundary conditions*

The boundary conditions of a quadrature element can be classified as two kinds: internal boundary conditions and external boundary conditions. For the external boundary conditions, the treatment has been discussed in the previous section. The compatibility conditions of two adjacent elements (see Fig. 2) require that at each node of an internal boundary

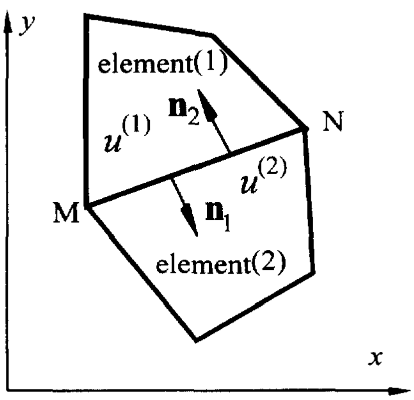

Fig. 2. Internal boundary of two adjacent elements.

$$u^{(1)}(x,y)|_{MN} = u^{(2)}(x,y)|_{MN} \tag{30}$$

$$\left.\frac{\partial u^{(1)}}{\partial \mathbf{n}_1}\right|_{MN} = -\left.\frac{\partial u^{(2)}}{\partial \mathbf{n}_2}\right|_{MN} \quad \text{or} \quad \left.\frac{\partial u^{(1)}}{\partial \mathbf{n}_1}\right|_{MN} + \left.\frac{\partial u^{(2)}}{\partial \mathbf{n}_2}\right|_{MN} = 0 \tag{31}$$

where $u^{(1)}(x,y)$, $u^{(2)}(x,y)$ are the functions defined in element 1 and element 2, respectively. MN is the common side of the two elements $\mathbf{n}_1$, $\mathbf{n}_2$ are the outward unit normal vectors on MN for the two elements, respectively. Hence, $\mathbf{n}_1 = -\mathbf{n}_2$. Let $\mathbf{n}_i = (\alpha_i, \beta_i)$, $(i = 1, 2)$, where α_i, β_i are the direction cosines of $\mathbf{n}_i$. Combining with eqn (9), eqn (31) can also be rewritten as

$$[\alpha_1, \beta_1][J_1]^{-1}\begin{Bmatrix} \dfrac{\partial u^{(1)}}{\partial \xi^{(1)}} \\ \dfrac{\partial u^{(1)}}{\partial \eta^{(1)}} \end{Bmatrix} + [\alpha_2, \beta_2][J_2]^{-1}\begin{Bmatrix} \dfrac{\partial u^{(2)}}{\partial \xi^{(2)}} \\ \dfrac{\partial u^{(2)}}{\partial \eta^{(2)}} \end{Bmatrix} = 0 \tag{32}$$

where $[J_1]^{-1}$, $[J_2]^{-1}$ are the inverse Jacobian matrices of element 1 and element 2, respectively. The differential quadrature format of eqn (32) can be implemented as what has been done to the fourth and fifth derivative terms on the left-side of eqn (7). It is noteworthy that the orientation of a quadrature element is arbitrary, indicating that at a node on the internal boundary $(\xi^{(1)}, \eta^{(1)})$ can be different from $(\xi^{(2)}, \eta^{(2)})$.

3.3. *On equations at element corners*

One of the technical details encountered in quadrature element analysis is the establishment of equations at the corners of a quadrature element. In the case of the second-order equation as addressed in the paper, one equation is set up at each node in an element. This is dictated by the need to have the same number of equations as unknowns. Generally, there are three possibilities for an element corner which are depicted in Fig. 3. In case (a) of Fig. 3, either of the two external boundary conditions (on AA_1 and AA_2) can be specified at the corner node A. Usually, the Dirichlet type boundary condition is chosen if there exists, since it offers accurate value of the function to be solved. In case (b), either the continuity condition (31) or the external boundary condition at A can be selected. As long as the mesh in each element is dense enough, say $N > 5$, it should have no significant effect on the accuracy of the solution. Case (c) in Fig. 3 represents the situation that A is an interior node shared by a number of elements. Since only one equation is needed at a corner point, the continuity condition on any one of the internal boundaries can be prescribed. Experience indicates that the choice of corner condition can be made on the basis of programming convenience and computational automation. Detailed discussion will be given in the first example below.

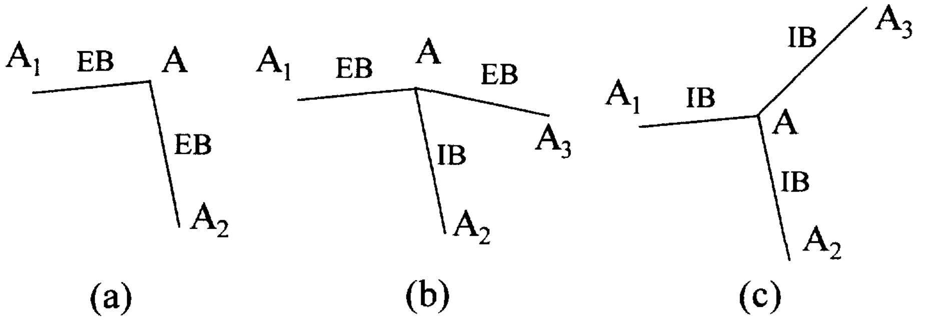

Fig. 3. Three possibilities of a corner point: EI—external boundary; IB—internal boundary.

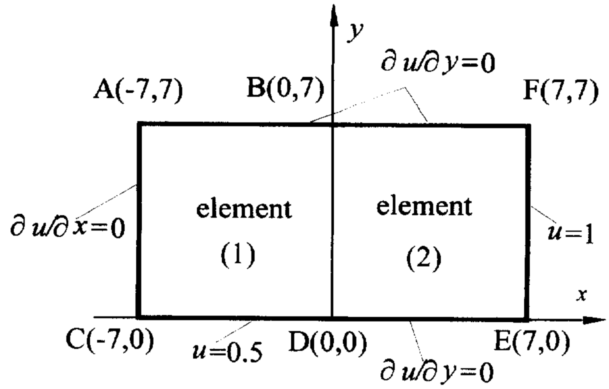

Fig. 4. Motz problem and the QEM.

4. EXAMPLES

4.1. *Example* 1 : *The Motz problem*

As a standard example with treatment of singularities involved in the solution of Laplace equation, the Motz problem has been treated by a number of authors. The most accurate results were obtained by Lefeber (1989) based on a series solution. The definition of the Motz problem (see Fig. 4) is as follows

$$\nabla^2 u = 0 \quad \text{in } \Omega \tag{33}$$

$$u = 0.5 \quad \text{on } y = 0, -7 \leqslant x \leqslant 0$$

$$u = 1.0 \quad \text{on } x = 7, 0 \leqslant y \leqslant 7$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \begin{cases} y = 0, & 0 \leqslant x \leqslant 7 \\ x = -7, & 0 \leqslant y \leqslant 7 \\ y = 7, & -7 \leqslant x \leqslant 7 \end{cases} \tag{34}$$

where n stands for the normal of the corresponding boundary. In this example, two quadrature elements are adopted to discretize the whole domain (see Fig. 4), according to the varying feature of the boundary conditions. The implementation of DQM in the whole domain has proven to yield completely wrong results. The reason is that the boundary condition on side $y = 0$ of the domain is inconsistent. In the QEM, the continuity of the boundary condition on one side of an element is maintained.

Comparison of the QEM results for different mesh size and those of the best analytical solution are shown in Fig. 5. It has been found that good results can be obtained by the QEM with 120 Degrees of Freedom (DOF) (mesh $8 \times 8 \times 2$). When the mesh density doubles, the solutions become spectacularly accurate, even for points near the singular origin $(0, 0)$.

To investigate the influence of selections of the corner condition on the solution of the Motz problem, the following two schemes are adopted at the corners of the two elements:

Scheme 1. u_A, u_B, u_C, u_D

A on AC Neumann	B on BD Continuity	C on AC Neumann	D on BD Continuity	E on DE Neumann	F on BF Neumann

Scheme 2. u'_A, u'_B, u'_C, u'_D

A on AB Neumann	B on AB Neumann	C on CD Dirichlet	D on CD Dirichlet	E on EF Dirichlet	F on EF Dirichlet

0.59330 0.59203 0.59136	0.59558 0.59424 0.59354	0.60231 0.60086 0.60012	0.61366 0.61207 0.61127	0.62985 0.62804 0.62717	0.65100 0.64894 0.64799	0.67662 0.67477 0.67377	0.70398 0.70613 0.70432	0.74262 0.74026 0.73920	0.78064 0.77870 0.77774	0.82166 0.82002 0.81917	0.86473 0.86341 0.86274	0.90915 0.90823 0.90776	0.95438 0.95391 0.95367	1.00000 1.00000 1.00000
0.59124 0.58987 0.58919	0.59339 0.59203 0.59134	0.60001 0.59858 0.59785	0.61126 0.60968 0.60889	0.62733 0.62557 0.62470	0.64838 0.64665 0.64549	0.67433 0.67239 0.67136	0.70529 0.70321 0.70214	0.74067 0.73842 0.73736	0.77931 0.77729 0.77629	0.82071 0.81899 0.81812	0.86408 0.86271 0.86202	0.90875 0.90780 0.90732	0.95419 0.95371 0.95346	1.00000 1.00000 1.00000
0.58464 0.58338 0.58275	0.58674 0.58545 0.58480	0.59312 0.59175 0.59105	0.60403 0.60250 0.60174	0.61979 0.61808 0.61722	0.64077 0.63885 0.63788	0.66716 0.66507 0.66399	0.69887 0.69662 0.69547	0.73525 0.73293 0.73178	0.77523 0.77306 0.77197	0.81782 0.81594 0.81500	0.86216 0.86068 0.85993	0.90760 0.90657 0.90606	0.95365 0.95314 0.95287	1.00000 1.00000 1.00000
0.57388 0.57279 0.57221	0.57581 0.57468 0.57410	0.58172 0.58048 0.57985	0.59194 0.59052 0.58980	0.60702 0.60537 0.60453	0.62770 0.62575 0.62476	0.65458 0.65232 0.65116	0.68772 0.68520 0.68391	0.72624 0.72362 0.72229	0.76857 0.76609 0.76484	0.81323 0.81109 0.81002	0.85918 0.85751 0.85668	0.90584 0.90470 0.90413	0.95284 0.95227 0.95198	1.00000 1.00000 1.00000
0.55927 0.55844 0.55797	0.56096 0.56004 0.55956	0.56605 0.56500 0.56447	0.57500 0.57375 0.57313	0.58866 0.58712 0.58635	0.60834 0.60638 0.60538	0.63555 0.63308 0.63180	0.67105 0.66811 0.66657	0.71346 0.71026 0.70864	0.75968 0.75667 0.75517	0.80740 0.80489 0.80363	0.85555 0.85364 0.85268	0.90375 0.90247 0.90183	0.95190 0.95125 0.95093	1.00000 1.00000 1.00000
0.54147 0.54092 0.54059	0.54279 0.54211 0.54176	0.54659 0.54580 0.54541	0.55342 0.55245 0.55197	0.56434 0.56305 0.56241	0.58135 0.57948 0.57856	0.60778 0.60511 0.60369	0.64721 0.64355 0.64156	0.69717 0.69278 0.69064	0.74950 0.74569 0.74381	0.80124 0.79827 0.79678	0.85191 0.84974 0.84864	0.90172 0.90030 0.89958	0.95100 0.95029 0.94993	1.00000 1.00000 1.00000
0.52181 0.52111 0.52091	0.52225 0.52171 0.52153	0.52417 0.52370 0.52349	0.52790 0.52735 0.52709	0.53420 0.53341 0.53303	0.54511 0.54365 0.54302	0.56625 0.56329 0.56195	0.61291 0.60669 0.60377	0.67950 0.67264 0.66954	0.74025 0.73556 0.73322	0.79622 0.79287 0.79115	0.84913 0.84675 0.84555	0.90023 0.89869 0.89792	0.95034 0.94958 0.94920	1.00000 1.00000 1.00000
0.50000 0.50000 0.50000	0.50000 0.50000 0.50000	0.50000 0.50000 0.50000	0.50000 0.50000 0.50000	0.50000 0.50000 0.50000	0.50000 0.50000 0.50000	0.50000 0.50000 0.50000	0.50000 0.50000 0.50000	0.66930 0.66056 0.65648	0.73597 0.73109 0.72847	0.79419 0.79072 0.78891	0.84808 0.84562 0.84437	0.89967 0.89810 0.89730	0.95009 0.94932 0.94893	1.00000 1.00000 1.00000

Fig. 5. Solution of the Motz problem: —8 × 8 × 2. —15 × 15 × 2. —Lefeber, 1989.

Table 1. Comparison of corner conditions

	Mesh							Lefeber, 1989
	3×3	5×5	7×7	9×9	11×11	13×13	15×15	
u_A	0.61345	0.59701	0.59416	0.59306	0.59254	0.59224	0.59205	0.591360
u'_A	0.59523	0.59567	0.59384	0.59294	0.59249	0.59221	0.59203	
$\|u_A - u'_A\|$	0.01822	0.00134	0.00032	0.00012	0.00005	0.00003	0.00002	
u_B	0.73109	0.71263	0.70850	0.70687	0.70607	0.70560	0.70531	0.704320
u'_B	0.78571	0.72385	0.71320	0.70944	0.70769	0.70673	0.70613	
$\|u_B - u'_B\|$	0.05462	0.01122	0.00470	0.00257	0.00162	0.00113	0.00082	
u_C	0.44538	0.48878	0.49531	0.49743	0.49837	0.49888	0.49918	0.5
u'_C	0.5	0.5	0.5	0.5	0.5	0.5	0.5	
$\|u_C - u'_C\|$	0.05462	0.01122	0.00469	0.00257	0.00163	0.00112	0.00082	
u_D	0.66387	0.59349	0.56898	0.55592	0.54765	0.54186	0.53752	0.5
u'_D	0.5	0.5	0.5	0.5	0.5	0.5	0.5	
$\|u_D - u'_D\|$	0.16387	0.09349	0.06898	0.05592	0.04765	0.04186	0.03752	

It is noted that the four corners of element 1 are quite representative in the sense that the pair of adjacent conditions for them are different from each other. Apart from the data at the four corners of element 1, there is virtually no difference in the results between the two schemes. The computed results on the four corners of element 1 are shown in Table 1.

It can be seen that the results from the two schemes are convergent. Apart from the function value at the singular point D, the function value of the two schemes are very close to each other with the refinement of mesh. As expected, the function value converges somewhat slowly at singular point D. Still, satisfactory results can be obtained with the increase of mesh density. At $11 \times 11 \times 2$, the relative error of function at D is reduced to less than 1%. Incidentally, function values at corners E and F are always exact due to the no flux condition at the top and bottom sides of element 2.

4.2. *Example 2: Torsion of a square shaft*

For comparison, the torsion of a prismatic shaft with square cross section, which has been treated using the DQM based on domain transformation (Zhong, 1996), is studied in this example (see Fig. 6). It also serves as an illustration that the quadrilateral element can be used to deal with triangular domain problems. The compatibility condition of an elastic prismatic shaft under torsion is the two dimensional Poisson equation (Timoshenko and Goodier, 1970) given as follows

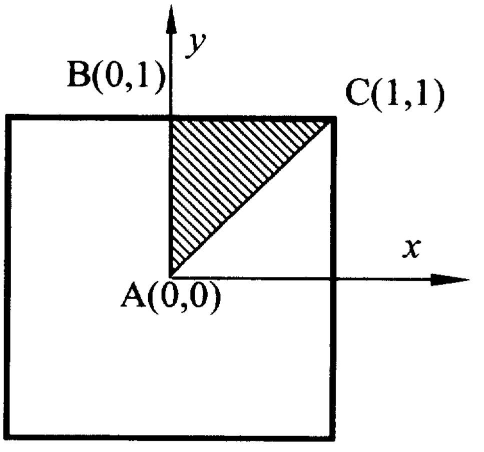

Fig. 6. Torsion of square shaft.

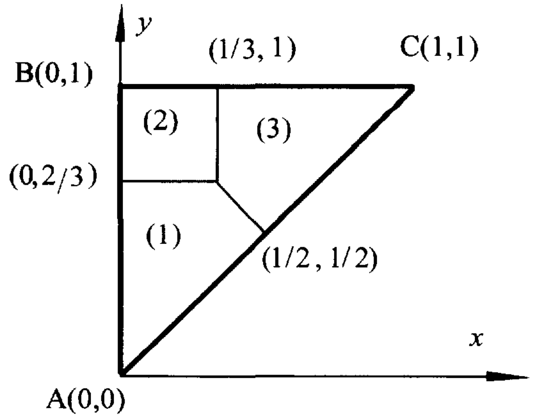

Fig. 7. Discretization of triangle.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -2G\kappa \tag{35}$$

where G is the shear modulus, κ the twist per unit length of the axis of the shaft, x and y are the space variables defined on the cross-section. For convenience, $G\kappa$ is taken as unity, i.e., $G\kappa = 1$. So eqn (35) becomes

$$\frac{\partial^2 \varphi}{\partial x} + \frac{\partial^2 \varphi}{\partial y^2} = -2 \tag{36}$$

Due to the symmetry of the problem, only one-eighth of the domain is considered here. The boundary conditions of the triangular domain are

$$\begin{aligned} \varphi &= 0, \quad \text{at side BC;} \\ \frac{\partial \varphi}{\partial n} &= 0, \quad \text{at sides AB and AC.} \end{aligned} \tag{37}$$

The domain is discretized into three quadrilateral elements (see Fig. 7). The selected computational results are shown in Table 2. It can be seen that the torsional function at the center of the cross-section φ_A and the maximum shear stress $\tau_{xz}|_B$ converge quickly to the exact solution with the refinement of the quadrature element mesh. To further study the present quadrilateral quadrature element, the results of the QEM and those of the

Table 2. Convergence of QEM for example 1

Element mesh	3×3	4×4	5×5	6×6	7×7	Exact solution
Total DOF	19	37	61	80	127	
φ_A	0.56103	0.60306	0.58884	0.58970	0.58923	0.58937
Err(%)	4.8	2.3	0.17	0.055	0.0085	—
$\tau_{xz}\|_B$	−1.3099	−1.3550	−1.3493	−1.3510	−1.3503	−1.3506
Err(%)	3.0	0.326	0.096	0.030	0.022	—

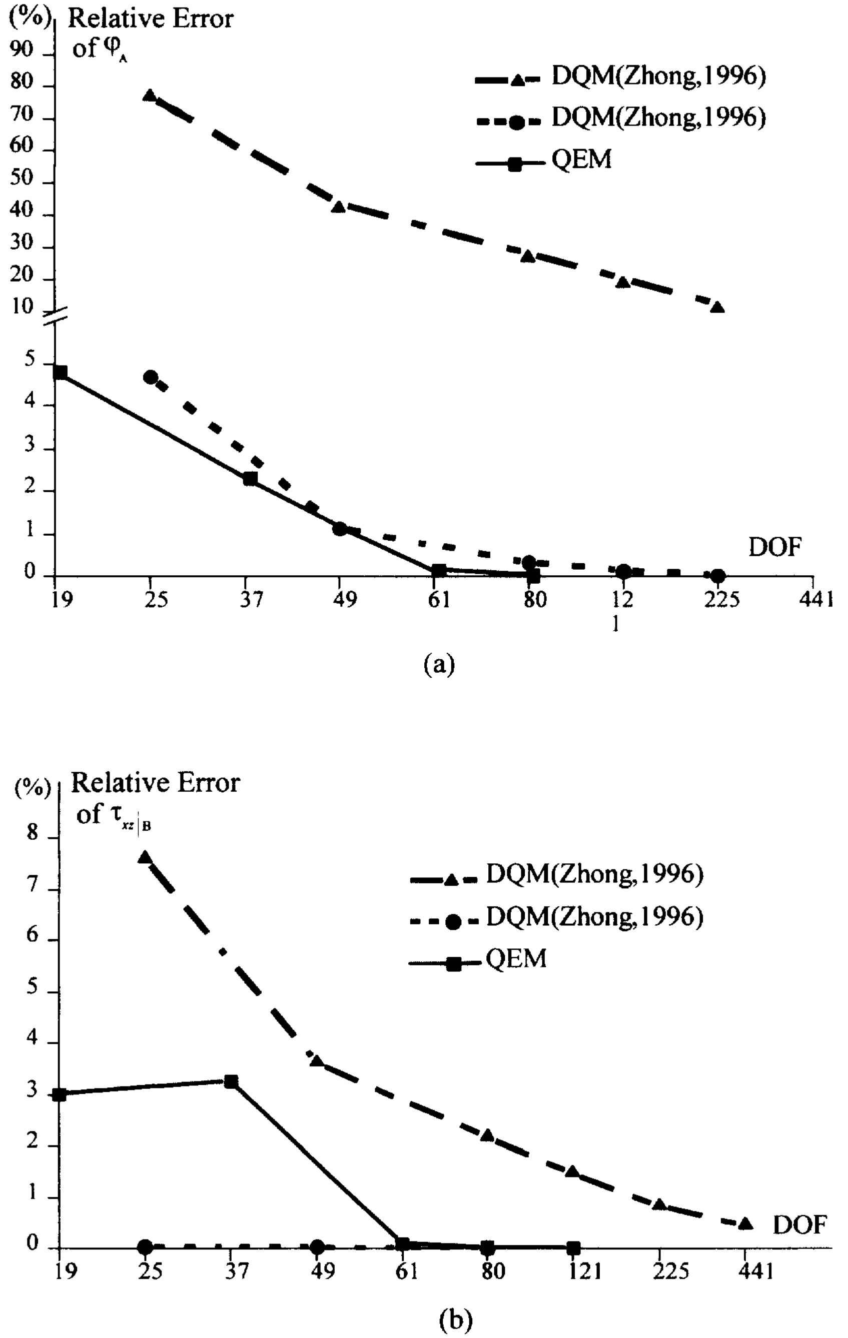

Fig. 8. Convergence comparison of DQM and QEM.

DQM for the same problem are compared in Fig. 8. Of the three available differential quadrature solutions based on three different domain transformations (Zhong, 1996), the selected two are the best and the worst, respectively. It can be observed that the present quadrature solution converges more quickly than the available differential solution for the torsional function at point A (see Fig. 8a). For the worst DQM solution, poor accuracy and slow convergence rate were produced. In the calculation of maximum shear stress, the available best differential quadrature transformation outperforms the present quadrature element mesh (see Fig. 8b). Nevertheless, the convergence speed of the present quadrature element solution is still satisfactory. With mere 19 DOF, the relative error of $\tau_{xz}|_B$ is around 3%. In addition, the treatment of singularity involved in the differential quadrature analysis is avoided in the present quadrature element analysis. Moreover, in view of the uneven performances of the differential quadrature solutions based on domain transformation, it is difficult to acquire a reliable solution in the first place since the transformation for the best differential quadrature is *a priori* unknown. Therefore, it may still be concluded that the overall performance of QEM is still better than that of the conventional DQM on the basis of domain transformation.

4.3. *Example* 3: *Temperature distribution*

In this example, the temperature distribution in an L-shaped living room (see Fig. 9) is studied by the QEM. The walls of the living-room are considered to be perfectly isolated so that the flux at the walls is zero, i.e., $\partial T/\partial n = 0$. The temperature of the window panes is assumed to be 10°C and the temperature of the chimney at the upper end of the living room is assumed to be 50°C. The temperature distribution T in the living room is the function that satisfies these boundary conditions and the Laplace equation, i.e.,

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \quad \text{in } \Omega \tag{38}$$

The domain is discretized into five elements (see Fig. 9) to treat the complex boundary conditions involved. The temperatures obtained by the QEM and the Boundary Element Method (BEM) (Hartmann, 1989) are compared on some selected points (see Fig. 10). It is noteworthy that the results of the BEM are not accurate enough and can only be used for reference. For this reason, significant discrepancy occurs between the QEM solution and the BEM solution on some grid points, especially the points on the boundary and near the boundary. From this example, it is seen that satisfactory results can be achieved by the QEM for a problem with complex geometry and complex boundary conditions, while the DQM is incapable of dealing with this situation.

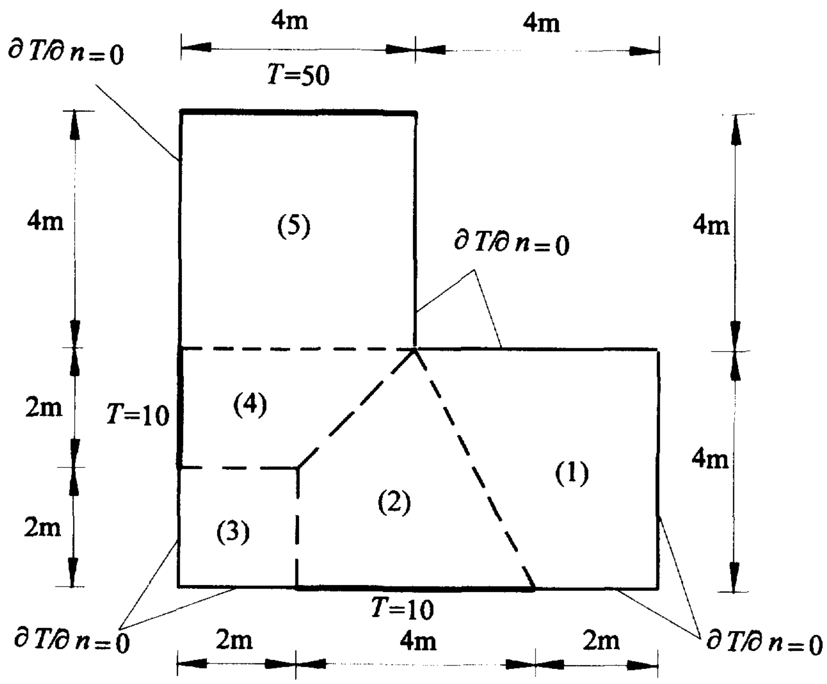

Fig. 9. Temperature distribution in an L-shape domain.

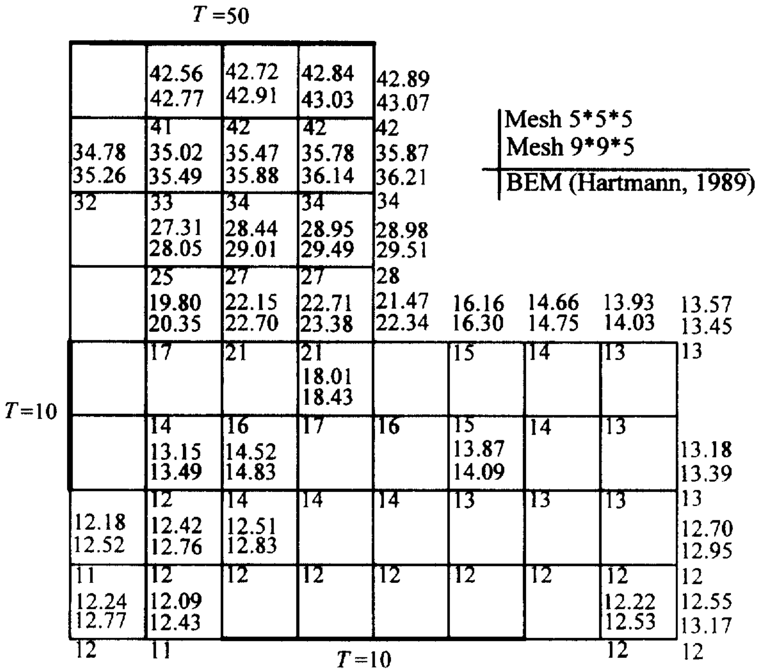

Fig. 10. Results for temperature distribution problem.

5. CONCLUDING REMARKS

The QEM offers an approach that is conceptually simple and mathematically straightforward since it combines the simplicity of the DQM and the versatility of the FEM. From the foregoing comparison of results for three numerical examples, it is seen that the proposed quadrilateral quadrature element is suitable for the solution of the two dimensional Poisson and Laplace equations on triangular and some irregular physical domains. It generally produces accurate results for comparable levels of computational effort. Even for problems defined on a simple geometric domain, the QEM also demonstrates its flexibility in the treatment of complex boundary conditions. The principal reason that the QEM may stand out as a new powerful tool is its easy formulation and coding contrasting to the conventional numerical tools such as the FEM and FDM.

It has been demonstrated that the QEM has great advantages over the DQM. Firstly, problems with complex geometry or complex boundary conditions can be perfectly solved, while the DQM is inapplicable in this situation. Secondly, singularity involved in the transformation of triangular domains can be avoided by the use of quadrilateral QEM elements. Thirdly, the QEM is generally more efficient than the DQM. It can achieve reliable and accurate results with less degrees of freedom.

Further work is clearly needed to extend the QEM to more practical problems. In addition, more element forms in addition to the quadrilateral element need be constructed.

REFERENCES

Bellman, R. E. and Casti, J. (1971) Differential quadrature and long term integration. *J. Math. Anal. Application* **34**, 235–238.

Hartmann, F. (1989) *Introduction to Boundary Elements—Theory and Applications*. Springer-Verlag, Berlin.

Jang, S. K., Bert, C. W. and Striz, A. G. (1989) Application of differential quadrature to static analysis of structural components. *International Journal of Numerical Methods in Engineering* **28**, 561–577.

Lefeber, D. (1989) *Solving Problems with Singularities Using Boundary Elements*. Computational Mechanics Publications, Southampton.

Shu, C. and Richards, B. E. (1992) Application of generalized differential quadrature to solve two-dimensional incompressible Navier-Stokes equations. *Int. J. Numer. Meth. in Fluids* **15**, 791–798.

Striz, A. G., Chen, W. and Bert, C. W. (1994) Static analysis of structures by the quadrature element method (QEM), *International Journal of Solids and Structures* **31**(20), 2807–2818.

Striz, A. G., Wang, X. and Bert, C. W. (1995) Harmonic differential quadrature method and applications to analysis of structural components. *Acta Mechanica* **111,** 85–94.

Timoshenko, S. P. and Goodier, J. N. (1970) *Theory of Elasticity*, 3rd edn. McGraw-Hill.

Zhong, H. Z. (1995) Quadrature element analysis of axisymmetric plates. Unpublished manuscript.

Zhong, H. Z. (1996) Elastic torsional analysis of prismatic shafts by differential quadrature method. *Int. J. Num. Method for Engng*, submitted.